FLOER HOMOLOGY AND EXISTENCE OF INCOMPRESSIBLE TORI IN HOMOLOGY SPHERES

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Abstract. We show that if a prime homology sphere has the same Floer homology as S^3 it does not contain any incompressible tori.

1. Introduction

Heegaard Floer homology, introduced by Ozsváth and Szabó in [OS1], has been the source of powerful techniques for the study three dimensional manifolds. It is shown in [OS3] that if a three-manifold obtained by a $1/n$-surgery on a knot $K \subset S^3$ has the Heegaard Floer homology of S^3, it should be homeomorphic to S^3 and the knot would be trivial. A generalization of this theorem is the subject of the following conjecture:

Conjecture. *If for a homology sphere Y, the Heegaard Floer homology package (including the groups and their grading) is isomorphic to the Heegaard Floer homology package of the sphere S^3, Y is homeomorphic to S^3.*

In this paper we will prove the following theorem, which has an interesting corollary supporting the above conjecture.

Theorem 1.1. *If K_i is a non-trivial knot in the homology sphere Y_i for $i = 1, 2$, and Y is obtained by splicing the complements of K_1 and K_2, the rank of $\widehat{\mathrm{HF}}(Y;\mathbb{Z}/2\mathbb{Z})$ will be bigger than one.*

Corollary 1.2. *If the prime homology sphere Y contains an incompressible torus, then the rank of Heegaard Floer homology group $\widehat{\mathrm{HF}}(Y)$ is bigger than one.*

Proof. Suppose otherwise that there is an incompressible torus T in a three-manifold Y with $\widehat{\mathrm{HF}}(Y;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, and denote the complement of T by $U_1 \cup U_2$. There are unique (up to isotopy) simple closed curves λ_i on T, $i = 1, 2$, such that the image of λ_i in the first homology of U_i is trivial. Since Y is a homology sphere, these two curves will generate the homology of T, and we may assume that they intersect each-other transversely in a single point. Attaching a disk to λ_2 in the boundary of U_1 and capping with a ball, we obtain a new homology sphere Y_1. Similarly, we can construct

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a homology sphere Y_2 from U_2. The curve λ_i will determine a knot K_i in Y_i, and Y is the three-manifold obtained by splicing the complement of K_1 to the complement of K_2. The previous theorem implies that at least one of the two knots, say K_1, should be the trivial knot in the corresponding three-manifold. This means that Y is obtained from Y_2 by a surgery on K_2, followed by taking connected sum with Y_1. Since T is an incompressible torus and K_1 is the un-knot in Y_1 , Y_1 is not S^3 and Y is not prime. □

The main ingredient of the proof is a gluing formula for the Heegaard Floer homology of the three-manifold obtained by splicing the complements of the two knots K_1 and K_2 which was established in [Ef2, Ef4]. This formula, along some other surgery results are quoted in section 2.

2. Splicing knot complements and surgery

Let $K \subset X$ denote a null-homologous knot inside a three-manifold X. Let $X_{p/q} = X_{p/q}(K)$ denote the result of p/q-surgery on K, and let $K_{p/q} \subset X_{p/q}$ be the knot in $X_{p/q}$ which is the core of the neighborhood replaced for $\mathrm{nd}(K) \subset X$ in forming $X_{p/q}$. Let

$$\mathbb{H}_{p/q} = \mathbb{H}_{p/q}(K) := \widehat{\mathrm{HFK}}(X_{p/q}, K_{p/q}; \mathbb{Z}/2\mathbb{Z}),$$

and denote the rank of this group by $h_{p/q} = h_{p/q}(K) = h(K_{p/q})$ and the rank of $\widehat{\mathrm{HF}}(X_{p/q}; \mathbb{Z}/2\mathbb{Z})$ by $y_{p/q} = y(X_{p/q})$.

In [Ef1] we established a surgery formula for Heegaard Floer homology. As a corollary, we have given a relatively simple formula for $\widehat{\mathrm{HFK}}(X_n(K), K_n; s)$ for $s \in \underline{\mathrm{Spin}}^c(X, K) = \mathbb{Z}$. The result was the following corollary:

Corollary 2.1. *Suppose that K is a knot inside a homology sphere X and $\mathbb{B}$ is the complex $\widehat{\mathrm{CF}}(X, \mathbb{Z}/2Z)$ together with the structure of a filtered chain complex induced by K. Let $\mathbb{B}\{\geq s\}$ denote the sub-complex generated by those generators whose relative Spin^c class in $\mathbb{Z} = \underline{\mathrm{Spin}}^c(X, K)$ is greater than or equal to $s \in \mathbb{Z}$. Then the homology group $\widehat{\mathrm{HFK}}(X_n(K), K_n; s; \mathbb{Z}/2\mathbb{Z})$ are isomorphic to the homology of the following chain complex*

$$C_n(s) = \Big(\mathbb{B}\{> s-n\} \xrightarrow{\subset} \mathbb{B} \xleftarrow{\supset} \mathbb{B}\{\geq -s\}\Big).$$

We have also shown that that the differential induced on $\mathbb{B}\{s\}$ from $\mathbb{B}$ may be assumed to be trivial when the coefficient ring is a field. We will assume that this is the case in the rest of this paper.

Let G be a directed graph with vertices $V(G) = \{v_1, ..., v_n\}$ and edges $E(G) = \{e_1, ..., e_m\}$, and let $i(v)$ and $t(e)$ in $V(G)$ be the initial and terminal point of the edge $e \in E(G)$. For $v \in V(G)$ let W_v be a given vector space over $\mathbb{Z}/2\mathbb{Z}$ and for $e \in E(G)$ let $f_e : W_{i(e)} \to W_{t(e)}$ be a given homomorphism

between vector spaces. This data will be encoded in a diagram with the vector space W_v placed at the vertex v of G and the homomorphism f_e placed on the directed edge e of G. Let $W = \oplus_{v \in V(G)} W_v$ and let $F_e : W \to W$ be the trivial extension of f_e to all of W. Let $d = F_{e_1} + ... + F_{e_m}$ be the sum of these maps. If $d \circ d = 0$, we will refer to the homology group $H_*(W, d)$ as the *homology of the diagram G*.

Under the identification of $\mathbb{H}_1(K)$ with the homology of the direct sum of mapping cones $C_1(s)$, two maps ϕ and $\overline{\phi}$ from $\mathbb{H}_1(K)$ to $\mathbb{H}_\infty(K)$ may be defined. Namely, we may identify ϕ and $\overline{\phi}$ as the maps induced by

$$\mathbb{B}\{> s-1\} = \mathbb{B}\{\geq s\} \longrightarrow \mathbb{B}\{s\} = \widehat{\mathrm{HFK}}(X, K; s), \quad \text{and}$$
$$\mathbb{B}\{\geq -s\} \longrightarrow \mathbb{B}\{-s\} \simeq \widehat{\mathrm{HFK}}(X, K; s)$$

respectively. For a fixed relative Spin^c class $s \in \underline{\mathrm{Spin}}^c(Y, K) = \mathbb{Z}$, the complexes $C_0(s)$ and $C_0(s-1)$ are both sub-complexes of $C_1(s)$ in a natural way. These two inclusions give rise to two maps $\psi, \overline{\psi} : \mathbb{H}_0(K) \to \mathbb{H}_1(K)$ so that $\overline{\psi} \circ \phi = \psi \circ \overline{\phi} = 0$. The following theorem was proved in [Ef2, Ef4]:

Theorem 2.2. *Suppose that (Y_1, K_1) and (Y_2, K_2) are two given knots inside three-manifolds Y_1 and Y_2 and let $\phi^i, \overline{\phi}^i, \psi^i$ and $\overline{\psi}^i$ be the corresponding maps for K_i, $i = 1, 2$. Let $\mathbb{C}\mathfrak{u} = \mathbb{C}\mathfrak{u}(K_1, K_2)$ denote the following cube of maps:*

$$\begin{tikzcd}
\mathbb{H}_{\infty,\infty} & & \mathbb{H}_{\infty,1} \arrow[ll, "I \otimes \phi^2"'] & \\
& \mathbb{H}_{1,\infty} \arrow[ul, "\phi^1 \oplus I"] & & \mathbb{H}_{1,1} \arrow[ll, "I \otimes \phi^2"'] \arrow[ul, "\phi^1 \oplus I"] \arrow[dd, "Id"] \\
\mathbb{H}_{0,0} \arrow[uu, "\overline{\eta}^1 \otimes \overline{\eta}^2"] \arrow[rr, "\psi^1 \otimes I"] \arrow[dr, "I \oplus \psi^2"'] & & \mathbb{H}_{1,0} \arrow[uu, "\overline{\phi}^1 \otimes \overline{\psi}^2"'] \arrow[dr, "I \oplus \psi^2"'] & \\
& \mathbb{H}_{0,1} \arrow[uu, "\overline{\psi}^1 \otimes \overline{\phi}^2"] \arrow[rr, "\psi^1 \otimes I"] & & \mathbb{H}_{1,1},
\end{tikzcd}$$

where $\mathbb{H}_{\star,\bullet} = \mathbb{H}_\star(K_1) \otimes \mathbb{H}_\bullet(K_2)$. Then the Heegaard Floer homology of the three-manifold Y, obtained by splicing knot complements $Y_1 - K_1$ and

$Y_2 - K_2$, is given by

$$\widehat{\mathrm{HF}}(Y;\mathbb{Z}/2\mathbb{Z}) = H_*(\mathbb{C}\mathfrak{u}),$$

where $H_*(\mathbb{C}\mathfrak{u})$ denotes the homology of the cube $\mathbb{C}\mathfrak{u}$.

The next result we would like to quote from [Ef3] is the following surgery formula which describes $\widehat{\mathrm{HF}}(X_{p/q},\mathbb{Z}/2\mathbb{Z})$:

Theorem 2.3. *With the above notation, $\widehat{\mathrm{HF}}(X_{p/q};\mathbb{Z}/2\mathbb{Z})$ may be obtained as the homology of the complex $(\mathbb{M}, d)$ such that*

$$\mathbb{M} = \Big(\bigoplus_{i=1}^{q}\mathbb{H}_\infty(i)\Big) \oplus \Big(\bigoplus_{i=1}^{p+q}\mathbb{H}_1(i)\Big) \oplus \Big(\bigoplus_{i=1}^{p}\mathbb{H}_0(i)\Big),$$

where each $\mathbb{H}_\bullet(i)$ is a copy of $\mathbb{H}_\bullet$. Moreover, the differential d is the sum of the following maps

$$\begin{aligned} &\phi^i : \mathbb{H}_\infty(i) \to \mathbb{H}_1(i), && \overline{\phi}^i : \mathbb{H}_\infty(i) \to \mathbb{H}_1(i+p), && i = 1, 2, ..., q \\ &\psi^j : \mathbb{H}_1(j+q) \to \mathbb{H}_0(j), && \overline{\phi}^j : \mathbb{H}_1(j) \to \mathbb{H}_0(j), && j = 1, 2, ..., p, \end{aligned}$$

where ϕ^i is the map ϕ corresponding to the copy $\mathbb{H}_\infty(i)$ of $\mathbb{H}_\infty$, etc..

We will need to compare these surgery formulas with surgery formulas of Ozsváth and Szabó ([OS3]). Let $\mathfrak{A}_s$ denote the sub-complex of $\mathrm{CFK}^\infty(Y, K)$ generated by those generators $[\mathbf{x}, i, j]$ such that $i(\mathbf{x})+i-j = s$ and such that $\max(i, j) = 0$. Here $i(\mathbf{x})$ denotes the relative Spin^c class in $\mathbb{Z} = \underline{\mathrm{Spin}^c}(Y, K)$ associated with the generator $\mathbf{x}$. The complex $\mathfrak{A}_s$ has two quotient complexes, one corresponding to $[\mathbf{x}, i, 0]$ with $i \le 0$ and $i(\mathbf{x}) = s - i \ge s$, and the other one corresponding to $[\mathbf{x}, 0, j]$ with $j \le 0$ and $i[\mathbf{x}] = s + j \le s$. The first one may be identified with $\mathbb{B}\{\ge s\}$ and the second one may be identified (via a duality isomorphism) with $\mathbb{B}\{\ge -s\}$. Both these quotient complexes are in fact sub-complexes of $\mathbb{B}$. The quotient map followed by the inclusion gives two maps h_s and v_s from the homology group $\mathbb{A}_s$ of $\mathfrak{A}_s$ to the homology group $\mathbb{H}$ of $\mathbb{B}$. One corollary of the result of Ozsváth and Szabó is the following:

Proposition 2.4. *If K is a knot inside the homology sphere X which has genus $g = g(K)$, the homology group $\widehat{\mathrm{HF}}(X_1(K))$ may be computed as the homology of the following diagram*

$$\begin{tikzcd} \mathbb{A}_{-g} \arrow[dr, "h_{-g}"] & & \mathbb{A}_{1-g} \arrow[dl, "v_{1-g}"'] \arrow[dr, "h_{1-g}"] & & \cdots & & \mathbb{A}_g \arrow[dl, "v_g"'] \\ & \mathbb{H} & & \mathbb{H} & \cdots\cdots & \mathbb{H} & \end{tikzcd}$$

Moreover, when $n > 2g$ is a given integer, for any integer $-n/2 \le s < n/2$ we may identify $\widehat{\mathrm{HF}}(X_n(K), [s])$ with $\mathbb{A}_s$, where $[s]$ denotes the class of s in $\mathbb{Z}/n\mathbb{Z}$.

We would like to examine the implications of the assumption $H_*(\mathbb{C}\mathfrak{u}) = \mathbb{Z}/2\mathbb{Z}$ in the rest of this paper.

3. Computing the homology of $\mathbb{H}$

The differential of the complex

$$\mathbb{C}\mathfrak{u} = \Big(\mathbb{H}_{\infty,\infty} \oplus \mathbb{H}_{1,1} \oplus \mathbb{H}_{1,0} \oplus \mathbb{H}_{0,1}\Big) \bigoplus \Big(\mathbb{H}_{1,\infty} \oplus \mathbb{H}_{\infty,1} \oplus \mathbb{H}_{1,1} \oplus \mathbb{H}_{0,0}\Big)$$

has the following block form $\mathfrak{D} = \begin{pmatrix} 0 & \mathfrak{D}_1 \\ \mathfrak{D}_2 & 0 \end{pmatrix}$, where $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are the following matrices respectively

$$\begin{pmatrix} I\otimes\phi_2 & \phi_1\otimes I & 0 & \overline{\eta}_1\otimes\overline{\eta}_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \psi_1\otimes I \\ 0 & 0 & 0 & I\otimes\psi_2 \end{pmatrix}, \quad \text{and} \begin{pmatrix} 0 & \phi_1\otimes I & \overline{\phi}_1\otimes\overline{\psi}_2 & 0 \\ 0 & I\otimes\phi_2 & 0 & \overline{\psi}_1\otimes\overline{\phi}_2 \\ 0 & I & I\otimes\psi_2 & \psi_1\otimes I \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The rank of the homology of $\mathbb{C}\mathfrak{u}$ may be computed as $|\mathbb{C}\mathfrak{u}| - 2.\mathrm{rnk}(\mathfrak{D}) = |\mathbb{C}\mathfrak{u}| - 2(\mathrm{rnk}(\mathfrak{D}_1)+\mathrm{rnk}(\mathfrak{D}_2))$. We would like to give upper estimates on the ranks of $\mathfrak{D}_1$ and $\mathfrak{D}_2$. The rank of $\mathfrak{D}_2$ is $|\mathbb{H}_{1,1}|+\mathrm{rnk}(\mathfrak{D}'_2)$, where $\mathfrak{D}'_2$ is the matrix

$$\mathfrak{D}'_2 = \begin{pmatrix} \overline{\phi}_1\otimes\overline{\psi}_2 + \phi_1\otimes\psi_2 & \eta_1\otimes I \\ I\otimes\eta_2 & \overline{\psi}_1\otimes\overline{\phi}_2 + \psi_1\otimes\phi_2 \end{pmatrix}$$

It is easy to see that if $x_i \in \mathrm{Ker}(\psi_i)$ and $y_i = \psi_i(z_i) \in \mathrm{Ker}(\overline{\phi}_i)$ are given, the vector

$$R(x_1,x_2,z_1,z_2) = \begin{pmatrix} \psi_1(z_1)\otimes x_2 \\ x_1\otimes\psi_2(z_2) \end{pmatrix}$$

is in the kernel of $\mathfrak{D}'_2$. Moreover, for arbitrary $x_i \in \mathbb{H}_0(K_i)$ we have the vector $R(x_1,x_2,x_1,x_2)$ in the kernel of $\mathfrak{D}'_2$. This implies that the size of the kernel of $\mathfrak{D}'_2$ is at least equal to

$$\begin{aligned} |\mathrm{Im}(\psi_1)|.|\mathrm{Ker}(\psi_2)| + |\mathrm{Ker}(\psi_1)|.|\mathrm{Im}(\psi_2)| + |\mathrm{Im}(\psi_1)|.|\mathrm{Im}(\psi_2)| \\ = |\mathbb{H}_{0,0}| - |\mathrm{Ker}(\psi_1)|.|\mathrm{Ker}(\psi_2)|. \end{aligned}$$

Consequently, the rank of $\mathfrak{D}_2$ is at most equal to

$$|\mathbb{H}_{0,1}| + |\mathbb{H}_{1,0}| - |\mathbb{H}_{0,0}| + |\mathrm{Ker}(\psi_1)|.|\mathrm{Ker}(\psi_2)|.$$

The rank of the first row of $\mathfrak{D}_1$ is at most $|\mathbb{H}_{0,0}|$, while the last two rows will have a rank which is at most $|\mathbb{H}_{\infty,\infty}| - |\mathrm{Ker}(\psi_1)|.|\mathrm{Ker}(\psi_2)|$. So the rank of $\mathfrak{D}$ will be at most equal to $|\mathbb{H}_{0,1}|+|\mathbb{H}_{1,0}|+|\mathbb{H}_{\infty,\infty}|-|\mathbb{H}_{0,0}|$. We have considered the row-rank of the matrix $\mathfrak{D}$ in the above considerations. If we consider the column rank and repeat the same argument we obtain that $\mathrm{rnk}(\mathfrak{D})$ is at most equal to $|\mathbb{H}_{\infty,1}| + |\mathbb{H}_{1,\infty}| - |\mathbb{H}_{\infty,\infty}| + |\mathbb{H}_{0,0}|$. This implies that the rank of the homology of $\mathbb{H}$ is at least equal to

$$||\mathbb{H}_{\infty,\infty}| + |\mathbb{H}_{0,1}| + |\mathbb{H}_{1,0}| - |\mathbb{H}_{\infty,1}| - |\mathbb{H}_{1,\infty}| - |\mathbb{H}_{0,0}||.$$

Since the term inside the absolute value function is an odd number, the rank is bigger than one, unless

$$|\mathbb{H}_{\infty,\infty}| + |\mathbb{H}_{0,1}| + |\mathbb{H}_{1,0}| - |\mathbb{H}_{\infty,1}| - |\mathbb{H}_{1,\infty}| - |\mathbb{H}_{0,0}| = \pm 1.$$

Without loss of generality, we will assume that this value is equal to $+1$. This implies that

$$\begin{aligned}\mathrm{rnk}(\mathfrak{D}_2) &= |\mathbb{H}_{0,1}| + |\mathbb{H}_{1,0}| - |\mathbb{H}_{0,0}| + |\mathrm{Ker}(\psi_1)|.|\mathrm{Ker}(\psi_2)|,\\ \mathrm{rnk}(\mathfrak{D}_1) &= |\mathbb{H}_{\infty,\infty}| + |\mathbb{H}_{0,0}| - |\mathrm{Ker}(\psi_1)|.|\mathrm{Ker}(\psi_2)|\end{aligned}$$

In an appropriate basis for $\mathbb{H}_\bullet(K_i)$, $i = 1, 2$, $\bullet \in \{0, 1, \infty\}$ the maps $\phi_i, \overline{\phi}_i, \psi_i$, and $\overline{\psi}_i$ have the following matrix block presentations:

$$\phi_i = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \ \overline{\psi}_i = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \ \overline{\phi}_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}, \text{ and } \ \psi_i = \begin{pmatrix} M_i & N_i \\ P_i & Q_i \end{pmatrix}.$$

If the second equality above is satisfied, the matrices D_i, $i = 1, 2$ and M_i, $i = 1, 2$ should be full-rank (if either of D_i is not full-rank, the first row of $\mathfrak{D}_1$ will not be of full rank, and if either of M_i is not full rank, the last column of $\mathfrak{D}_1$ will not be of full rank). Moreover, D_i will be a $t_i \times s_i$ matrix with $t_i \leq s_i$ and M_i will be a $r_i \times t_i$ matrix with $t_i \leq r_i$. Having in mind that the matrix multiplications $\overline{\phi}_i.\psi_i$ give zero for $i = 1, 2$, we may assume that by further modification of the basis for the vector spaces $\mathbb{H}_\bullet(K_i)$, the matrix presentations of these maps are given by:

$$\phi_i = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \overline{\phi}_i = \begin{pmatrix} 0 & a_i & b_i & 0 \\ 0 & c_i & d_i & 0 \\ 0 & 0 & 0 & I \end{pmatrix},$$

$$\overline{\psi}_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}^t, \quad \text{and} \quad \psi_i = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & m_i^t & p_i^t & 0 \\ 0 & n_i^t & q_i^t & 0 \end{pmatrix}^t.$$

The matrices a_i, b_i, c_i and d_i are of sizes $t_i \times (r_i - t_i), t_i \times (s_i - t_i), (r_i - t_i) \times (r_i - t_i)$ and $(r_i - t_i) \times (s_i - t_i)$ respectively, while the matrices m_i, n_i, p_i and q_i are of sizes $(r_i - t_i) \times (s_i - t_i), (r_i - t_i) \times t_i, (s_i - t_i) \times (s_i - t_i)$ and $(s_i - t_i) \times t_i$ respectively.

A vector in this block decomposition of the form

$$\left(\left(\begin{bmatrix} h_1 \\ u_1 \\ v_1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} h_2 \\ w_2 \\ 0 \end{bmatrix} \right)^t + \left(\begin{bmatrix} h_1 \\ w_1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} h_2 \\ u_2 \\ v_2 \\ 0 \end{bmatrix} \right)^t \right)^t$$

with $m_i w_i = p_i w_i = 0$ is in the kernel of $\mathfrak{D}_2'$ if and only if

$$(c_1 u_1 + d_1 v_1) \otimes w_2 = w_1 \otimes (c_2 u_2 + d_2 v_2) = 0.$$

Any non-trivial solution to these equations gives an element in the kernel which is not of the form considered previously. This quickly implies that at least for one of the two knots both $\beta_i = (c_i \quad d_i)$ and $\gamma_i^t = (m_i^t \quad p_i^t)$ are

full-rank. Thus $\alpha_i = (a_i \quad b_i)$ should be a multiple of $\beta_i = (c_i \quad d_i)$ i.e. there is some matrix ϵ_i such that $\alpha_i = \epsilon_i.\beta_i$. Similarly $\theta_i^t = (n_i^t \quad q_i^t)$ is a multiple of $\gamma_i^t = (m_i^t \quad p_i^t)$, i.e. there some matrix δ_i such that $\theta_i = \gamma_i.\delta_i$. As a result, there are full-rank matrices Φ_i, Γ_i, Ψ_i and Υ_i such that, in an appropriate basis, the matrix block forms for the maps $\phi_i, \overline{\phi}_i, \psi_i$, and $\overline{\psi}_i$ are given by

$$\begin{pmatrix} I & 0 & 0 \\ 0 & \Phi_i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Gamma_i & 0 \\ 0 & 0 & I \end{pmatrix}, \begin{pmatrix} I & 0 & 0 \\ 0 & \Psi_i & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Upsilon_i & 0 \\ 0 & 0 & I \end{pmatrix},$$

respectively. Below, we will study the properties of a knot K inside a homology sphere X with the corresponding maps $\phi, \overline{\phi}, \psi$, and $\overline{\psi}$ such that in appropriate basis for $\mathbb{H}_\bullet(K)$, the maps $\phi, \overline{\phi}, \overline{\psi}$ and ψ are of the above form for some full-rank matrices Φ, Ψ, Γ and Υ with the property that $\Phi.\Psi = \Gamma.\Upsilon = 0$. As a first step, combining with theorem 2.3 we have the following corollary:

Corollary 3.1. *If (X, K) is as above then*

$$y_{p/q}(X) := |\widehat{\mathrm{HF}}(X_{p/q})| = p(h_1(K) - h_0(K)) + q(h_1(K) - h_\infty(K)).$$

Proof. Suppose that in the basis where the maps have the above presentations, the groups $\mathbb{H}_\bullet(K)$ have the following decompositions:

$$\mathbb{H}_\infty(K) = A \oplus M \oplus B, \quad \mathbb{H}_1(K) = A \oplus H \oplus B, \text{ and } \mathbb{H}_0(K) = A \oplus L \oplus B.$$

Because of the form of the maps in this basis, the groups A and B do not have any contribution to the homology of the complex in theorem 2.3. Since the maps $\Phi, \Gamma : M \to H$ are injective and the maps $\Psi, \Upsilon : H \to L$ are surjective, it is implied that the contribution of the part corresponding to M, H, and L is $p(|H| - |L|) + q(|H| - |M|)$. This implies the claim. $\square$

The asymptotic behavior of $y_n = y_{n/1}$ is like $ny_\infty + c$, for some constant c. As a result $y_\infty = h_1 - h_0$. Denote $H_*(\mathbb{B}\{\bullet\})$ by $\mathbb{H}\{\bullet\}$.

Lemma 3.2. *If (X, K) is as above, the map $\imath : \mathbb{H}\{\geq s\} \to \mathbb{H}$, induced by inclusion, is surjective for $s \leq 0$.*

Proof. Since $y_n = ny_\infty + c$ for all n, by proposition 2.4 the constant c may be computed as $\sum_s(|\mathbb{A}_s| - |\mathbb{H}|)$. Thus $y_1 = \left(\sum_{s=-g}^{g} |\mathbb{A}_s|\right) - 2g|\mathbb{H}|$, where $g = g(K)$ is the genus of K. It is implied that $\mathrm{Im}(h_s) + \mathrm{Im}(v_{s+1})$ is all of $\mathbb{H}$ for each s. In particular, for $s = -1$ the sum of images of $\mathbb{H}\{\geq 1\}$ and $\mathbb{H}\{\geq 0\}$ under the map induced by inclusion is all of $\mathbb{H}$. Since the image of the first one is included in the image of the second, $\imath : \mathbb{H}(\geq 0) \to \mathbb{H}$ is surjective. $\square$

For any element x in the complex $\mathbb{B}$ there is a largest value $s = \tau(x) \in \mathbb{Z}$ and another element $y \in \mathbb{B}$ such that $x + dy \in \mathbb{B}\{\geq s\}$. The function τ is constant on the co-sets of $\mathrm{Im}(d)$. If $[x] \in \mathbb{H}$ is represented by a closed

generator $x \in \mathbb{B}\{\geq \tau(x)\}$, since $\tau(x) \geq 0$ we conclude that $(x, 0, x)$ in the mapping cone

$$\mathbb{H}\{\geq \tau(x)\} \xrightarrow{\imath} \mathbb{H} \xleftarrow{\imath} \mathbb{H}\{\geq -\tau(x)\}$$

will give a generator for $\mathbb{H}_1(K, \tau(x))$ and its image under ϕ in $\mathbb{H}_\infty(K, \tau(x))$ is the class of x in the quotient complex $\mathbb{H}\{\tau(x)\}$ of $\mathbb{B}\{\geq \tau(x)\}$. Moreover, if $z = dx$ for some $x \in \mathbb{B}$ and $x \in \mathbb{B}\{\geq \tau(x)\}$ and $z \in \mathbb{B}\{\geq \tau(z)\}$, then $\tau(z) > \tau(x)$ and $(z, 0, 0)$ in

$$\mathbb{H}\{\geq \tau(z)\} \xrightarrow{\imath} \mathbb{H} \xleftarrow{\imath} \mathbb{H}\{\geq -\tau(z)\}$$

represents a non-zero class in the homology of this complex whose image under ϕ is the class z in the quotient complex $\mathbb{H}\{\tau(z)\}$ of $\mathbb{B}\{\geq \tau(z)\}$.
The above two observation identify a subset of $\mathrm{Im}(\phi)$ which is in correspondence with $\mathrm{Ker}(d_* : \mathbb{H}_\infty \to \mathbb{H}_\infty)$. The size of this subset is $(y_\infty + h_\infty)/2$. We have $h_1 + h_\infty - 2\mathrm{rnk}(\phi) = h_0 = h_1 - y_\infty$. This means that what we have identified in $\mathrm{Im}(\phi)$ is in fact all of the image of ϕ.
For the element $x \in \mathbb{H}\{\geq 0\}$ in

$$\mathbb{H}_1(K, 0) = H_*\Big(\mathbb{H}\{\geq 0\} \xrightarrow{i} \mathbb{H} \xleftarrow{\imath} \mathbb{H}\{\geq 0\}\Big),$$

the closed class $(x, 0, x)$ is mapped to $[x] \in \mathbb{H}_\infty(0)$ by ϕ and to $\tau[x]$ by $\overline{\phi}$, where τ is the duality map discussed earlier. This is in fact all of the image for both maps ϕ and $\overline{\phi}$ by our previous observation. Since $\mathrm{Im}(\phi)$+$\mathrm{Im}(\overline{\phi})$ should cover all of $\mathbb{H}_\infty(K, 0)$, which is an odd dimensional vector space, there should be some closed element x as above such that $[x] = \tau[x]$. Let Q_1 be the sub-complex of $\mathbb{H}_1(K, 0)$ generated by the element above and denote by Q_∞ the image of this element under ϕ and $\overline{\phi}$. Clearly, Q_1 is disjoint from $\mathrm{Im}(\psi)$+$\mathrm{Im}(\overline{\psi})$. So Q_1 and Q_∞ determine a component of the cube $\mathbb{C}\mathfrak{u}$ which may be identified with the following diagram of maps

$$\begin{tikzcd}
\mathbb{H}_\infty(K_1) \otimes Q_\infty & & \mathbb{H}_\infty(K_1) \otimes Q_1 \arrow[ll, "I"'] & \\
& \mathbb{H}_1(K_1) \otimes Q_\infty \arrow[ul, "\phi 1"'] & & \mathbb{H}_1(K_1) \otimes Q_1 \arrow[ll, "I"'] \arrow[ul, "\phi 1"'] \arrow[d, "I"] \\
& \mathbb{H}_0(K_1) \otimes Q_1 \arrow[u, "\overline{\psi}_1 \otimes \tau"] \arrow[rr, "\psi^1"] & & \mathbb{H}_1(K_1) \otimes Q_1.
\end{tikzcd}$$

It is easy to check that the homology of this complex is the same as the homology of the mapping cone

$$\psi^1 \otimes I + \overline{\psi}^1 \otimes \tau : \mathbb{H}_0(K_1) \otimes Q \to \mathbb{H}_1(K_1) \otimes Q.$$

The rank of the homology of $\mathbb{H}$ is thus greater than or equal to $(h_1(K_1) - h_0(K_1))|Q_2|$ (We already know that $h_1(K_i) \geq h_0(K_i)$). Consequently, $q_2 = |Q_2| = 1$ and $h_1(K_1) = 1 + h_0(K_1)$. Similarly we have $h_1(K_2) = 1 + h_0(K_2)$. We remind the reader that this is the same as claiming $y_\infty(Y_i) = 1$ for $i = 1, 2$.

We have seen before that the rank of the homology of $\mathbb{H}$ is at least equal to

$$|\chi(\mathbb{C}\mathfrak{u})| = ||\mathbb{H}_{\infty,\infty}| + |\mathbb{H}_{0,1}| + |\mathbb{H}_{1,0}| - |\mathbb{H}_{\infty,1}| - |\mathbb{H}_{1,\infty}| - |\mathbb{H}_{0,0}||.$$

Since we know that $h_1(K_i) = h_0(K_i) + 1$ this quantity may be written as

$$|\chi(\mathbb{C}\mathfrak{u})| = |(h_1(K_1) - h_\infty(K_1))(h_1(K_2) - h_\infty(K_2)) - 1|.$$

Note that $h_1(K_i) - h_\infty(K_i)$ is even. For $\chi(\mathbb{C}\mathfrak{u})$ to be ± 1 the only possibility is the assumption that for one of the two knots, say for K_1, we have $h_1(K_1) = h_\infty(K_1)$. This implies that the values of r_1, s_1 are equal to $t_1 + 1$ and t_1 respectively, i.e. the size of matrices Φ_1 and Ψ_1 should be 1×1 and 1×0 respectively. Since both Φ_1 and Γ_1 are full-rank, we should have $\Phi_1 = \Gamma_1 = 1$. We conclude that $y_\infty = y_1 = 1$ while $y_0 = 2$.

Lemma 3.3. *In the above situation, $d_* : \mathbb{H}(s) \to \mathbb{H}\{> s\}$ is surjective for $s \geq 0$ and has one dimensional co-kernel for $s < 0$.*

Proof. The complex giving $\mathbb{A}_s$ has an odd number of generators, so $\mathbb{A}_s$ is odd dimensional and non-trivial for each s. For

$$y_1 = \Big(\sum_{s=-g}^{g} |\mathbb{A}_s|\Big) - 2g|\mathbb{H}| = \Big(\sum_{s=-g}^{g} |\mathbb{A}_s|\Big) - 2g$$

to be equal to 1, we should have $\mathbb{A}_s = \mathbb{Z}/2\mathbb{Z}$ for all s. Note that $\mathbb{A}_s$ is the homology of the mapping cone

$$\mathbb{H}\{> s\} \xleftarrow{d_*} \mathbb{H}(s) \xrightarrow{d_* \circ \tau} \mathbb{H}\{> -s\}$$

and for $s < 0$ the term on the left-hand-side contains a generator which survives in the homology $\mathbb{H}$ of $\mathbb{B}$ (so is not in the image of d_*). Thus the lemma follows for $s \neq 0$ once we note that the total homology of this mapping cone is $\mathbb{Z}/2\mathbb{Z}$. When $s = 0$, if d_* is not surjective, we will have non-exact elements of this mapping cone both on the right-hand-side and the left-hand-side. $\square$

We can now finish the proof of the following theorem:

Theorem 3.4. *If K_i is a non-trivial knot in a homology sphere Y_i, $i = 1, 2$, and Y is obtained by splicing the complements of K_1 and K_2, the rank of $\widehat{\mathrm{HF}}(Y)$ will be bigger than one.*

Proof. We have seen that if the rank of $\widehat{\mathrm{HF}}(Y)$ is non-trivial, we obtain a knot K inside a homology sphere X such that the corresponding homology

groups $\mathbb{A}_s$ are all isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and consequently the above lemma is true.

Let $g = g(K) > 0$ and note that by lemma 3.2, the map $d_* : \mathbb{H}\{-g\} \to \mathbb{H}\{> -g\}$ is injective. Denote the image of d_* by E and its domain by E'. Let $E^{\perp}$ be the co-kernel of d_* in $\mathbb{H}\{> -g\}$, which should be isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Note that $\mathbb{A}_{1-g}$ is isomorphic to the homology of the mapping cone

$$\mathbb{H}\{> -g\} = E \oplus E^{\perp} \xrightarrow{d_* \circ \tau \circ q} E' = \mathbb{H}\{-g\} \simeq \mathbb{H}\{g\},$$

where q is the quotient map from $\mathbb{H}\{> -g\}$ to $\mathbb{H}\{1-g\}$ and τ is the isomorphism between $\mathbb{H}\{1-g\}$ and $\mathbb{H}\{g-1\}$. We conclude that the map $d'_* = d_* \circ \tau \circ q$ should be surjective.
On the other hand, if d_∞ denotes the differential of the complex $\mathrm{CFK}^\infty(X,K)$, from $d_\infty^2 = 0$ and the fact that $\mathbb{H}\{s\} = 0$ for $s < -g$ we may easily conclude that $d'_* \circ d_*$ is trivial on $E' = \mathbb{H}\{-g\}$. This means that d'_* maps $E^{\perp}$ surjectively to E'. This would mean that the homology of $\mathbb{A}_{1-g}$ contains E as a subspace, i.e. E and E' should be isomorphic to $\mathbb{Z}/2\mathbb{Z}$.
Similarly, the vanishing of d_∞^2 and the vanishing of d_* on $E^{\perp}$ imply that $d_* \circ d'_*$ is trivial on $E^{\perp}$. But $d'_*(E^{\perp}) = E'$ and $d_*(E') = E$ which is a contradiction. This implies that $g(K)$ can not be positive, i.e. K is trivial □

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